

TOPOLOGICALLY 4-CHROMATIC GRAPHS AND SIGNATURES OF ODD CYCLES

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ABSTRACT. We investigate group-theoretic “signatures” of odd cycles of a graph, and their connections to topological obstructions to 3-colourability. In the case of signatures derived from free groups, we prove that the existence of an odd cycle with trivial signature is equivalent to having the coindex of the hom-complex at least 2 (which implies that the chromatic number is at least 4). In the case of signatures derived from elementary abelian 2-groups we prove that the existence of an odd cycle with trivial signature is a sufficient condition for having the index of the hom-complex at least 2 (which again implies that the chromatic number is at least 4).

Keywords: Graph colourings, homomorphisms, Hom complexes, free groups

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1. INTRODUCTION

This paper is motivated by so-called “topological bounds” on the chromatic number of a graph:

$$(1) \quad \chi(H) \geq \text{ind}(\text{Hom}(K_2, H)) + 2 \geq \text{coind}(\text{Hom}(K_2, H)) + 2.$$

Here, $\text{Hom}(K_2, H)$ is a “hom-complex” which can be viewed both as a \mathbb{Z}_2 -poset and as the geometric realisation of its order complex. Its index $\text{ind}(\text{Hom}(K_2, H))$ and coindex $\text{coind}(\text{Hom}(K_2, H))$ will be defined in the next section.

The bounds (1) have been useful in determining chromatic numbers for various classes of graphs. However for general graphs, the index and coindex are not known to be computable. In contrast, the chromatic number is in NP. The computational aspects of such topological invariants are now being investigated (see [2, 1]). In this paper we focus on the case when the bounds give a chromatic number of at least 4. We present an algebraic approach.

For a graph H , let $A(H)$ denote the set of its arcs. That is, for $[u, v] \in E(H)$, $A(H)$ contains the two arcs (u, v) and (v, u) . Let \mathcal{V} be a variety of groups (in the sense of universal algebra: a class of groups defined by a set of identities). Let $\mathcal{F}_{\mathcal{V}}(A(H))$ be the free group in \mathcal{V} generated by the elements of $A(H)$. We define the congruence θ on $\mathcal{F}_{\mathcal{V}}(A(H))$ by the relations

$$(a, b)(c, b)^{-1}(c, d)(a, d)^{-1} \sim_{\theta} 1$$

for all 4-cycles a, b, c, d of H . The group $\mathcal{G}_{\mathcal{V}}(H)$ is defined as the quotient $\mathcal{F}_{\mathcal{V}}(A(H))/\theta$. Let C_n denote the cycle with vertex-set $\mathbb{Z}_n = \{0, \dots, n-1\}$ and edges $[i, i+1]$, $i \in \mathbb{Z}_n$. If n is odd and $f : C_n \rightarrow H$ is a homomorphism, we define the \mathcal{V} -signature

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$\sigma_{\mathcal{V}}(f)$ of f as

$$\sigma_{\mathcal{V}}(f) = \prod_{i=0}^{n-1} (f(2i), f(2i+1)) \cdot (f(2i+2), f(2i+1))^{-1},$$

where the indices are taken modulo $2n+1$ and the product is developed left to right: $\prod_{i=0}^k x_i = x_0 x_1 \cdots x_k$ rather than $x_k x_{k-1} \cdots x_0$.

We will consider two varieties of groups and related signatures: the variety \mathcal{V}_1 of all groups and the variety \mathcal{V}_2 of elementary abelian 2-groups (where we use additive notation). We let $\sigma_1(f)$ and $\sigma_2(f)$ denote respectively $\sigma_{\mathcal{V}_1}(f)$ and $\sigma_{\mathcal{V}_2}(f)$. We prove the following results.

Theorem 1. *Let H be a graph. Then $\text{coind}(\text{Hom}(K_2, H)) \geq 2$ if and only if for some odd n , there exists a homomorphism $f : C_n \rightarrow H$ such that $\sigma_1(f) = 1$.*

Theorem 2. *Let H be a graph. If for some odd n there exists a homomorphism $f : C_n \rightarrow H$ such that $\sigma_2(f) = 0$, then $\text{ind}(\text{Hom}(K_2, H)) \geq 2$.*

We will show that the existence of an odd n and a homomorphism $f : C_n \rightarrow H$ such that $\sigma_2(f) = 0$ can be decided in polynomial time. Therefore if the converse of Theorem 2 holds, then the question as to whether a graph H satisfies $\text{ind}(\text{Hom}(K_2, H)) \geq 2$ can be decided in polynomial time. In contrast, Theorem 1 provides a necessary and sufficient algebraic condition for a graph H to satisfy $\text{coind}(\text{Hom}(K_2, H)) \geq 2$, but it is not clear whether this condition can be decided at all, let alone in polynomial time. In fact, as pointed out by Zimmerman [8], it is not clear whether the word problem in $\mathcal{G}_{\mathcal{V}_1}(H)$ is always decidable.

2. PRELIMINARIES

In this section we introduce the terminology necessary to define the index and coindex of hom-complexes, and characterise them in ways that will allow us to prove Theorems 1 and 2.

2.1. Topology. The *hom-complex* $\text{Hom}(K_2, H)$ of H is the set with elements (A, B) such that A, B are nonempty subsets of H and every element of A is joined by an edge of H to every element of B . Here, K_2 denotes the complete graph on two vertices 0 and 1; the name “hom”-complex is derived from the fact that if $(A, B) \in \text{Hom}(K_2, H)$, then for any $a \in A$ and $b \in B$, there is a homomorphism f of K_2 to H defined by $f(0) = a$ and $f(1) = b$.

We view $\text{Hom}(K_2, H)$ primarily as a \mathbb{Z}_2 -poset, that is a poset with a fixed-point free automorphism of order two (denoted $-$). The order relation on $\text{Hom}(K_2, H)$ is coordinatewise inclusion, and the \mathbb{Z}_2 involution is given by $-(A, B) = (B, A)$. A \mathbb{Z}_2 -map between \mathbb{Z}_2 -posets P and Q is an order-preserving map $f : P \rightarrow Q$ such that $f(-x) = -f(x)$.

Any poset P can also be viewed as a simplicial complex, by viewing chains as simplices. The *geometric realization* of P is the topological space $\Delta P \subseteq \mathbb{R}^P$ induced by the functions $f : P \rightarrow [0, 1]$ whose support (i.e., set of elements with nonzero image) is a chain in P , and which satisfy $\sum \{f(x) : x \in P\} = 1$. If P is a \mathbb{Z}_2 -poset, then ΔP is a \mathbb{Z}_2 -space, that is, a topological space with a fixed-point free homeomorphism of order 2. A \mathbb{Z}_2 -map between \mathbb{Z}_2 -spaces X and Y is a continuous map $f : X \rightarrow Y$ such that $f(-x) = -f(x)$.

The index $\text{ind}(X)$ and coindex $\text{coind}(X)$ of a \mathbb{Z}_2 -space X are defined in terms of the unit sphere $S_n \subseteq \mathbb{R}^{n+1}$, viewed as a \mathbb{Z}_2 -space:

- $\text{ind}(X)$ is the smallest n such that X admits a \mathbb{Z}_2 -map to S_n ;
- $\text{coind}(X)$ is the largest n such that S_n admits a \mathbb{Z}_2 -map to X .

The fact that $\text{ind}(S_n) = \text{coind}(S_n) = n$ is not trivial, but is a restatement of the Borsuk-Ulam theorem.

For a \mathbb{Z}_2 -poset P , we write $\text{ind}(P)$ and $\text{coind}(P)$ respectively for $\text{ind}(\Delta P)$ and $\text{coind}(\Delta P)$. Now consider the $(2n+2)$ -element \mathbb{Z}_2 -poset Q_n , with elements $\{\pm 0, \dots, \pm n\}$ ordered by the relation $\{+i, -i\} < \{+j, -j\}$ (in Q_n) when $i < j$ (in \mathbb{N}). Then Q_n is the face-poset of the cross-polytope of dimension n , and therefore ΔQ_n is \mathbb{Z}_2 -homeomorphic to S_n . This correspondence can be used to characterise the index and the coindex of a \mathbb{Z}_2 -poset in terms of order-preserving \mathbb{Z}_2 -maps.

The *barycentric subdivision* of a poset P is the poset $S(P)$ whose elements are the chains of P , ordered by inclusion. Note that when P is a \mathbb{Z}_2 -poset, $S(P)$ is also a \mathbb{Z}_2 -poset. The exponential notation is used to denote iterated barycentric subdivisions. By simplicial approximation, the following holds for any \mathbb{Z}_2 -poset P :

- $\text{ind}(P)$ is the smallest n such that for some m , $S^m(P)$ admits a \mathbb{Z}_2 -map to Q_n ;
- $\text{coind}(P)$ is the largest n such that for some m , $S^m(Q_n)$ admits a \mathbb{Z}_2 -map to P .

2.2. Graph theory. For a graph H , the characterisation of $\text{ind}(\text{Hom}(K_2, H))$ given just above will be sufficient to prove Theorem 2 in Section 4 below. For a proof of Theorem 1, we rely on a further characterisation of $\text{coind}(\text{Hom}(K_2, H))$ in terms of graph homomorphisms.

The *categorical product* of two graphs G and G' is the graph $G \times G'$ defined by

$$\begin{aligned} V(G \times G') &= V(G) \times V(G'), \\ E(G \times G') &= \{[(u, u'), (v, v')] : [u, v] \in E(G) \text{ and } [u', v'] \in E(G')\}. \end{aligned}$$

For $q \in \mathbb{N}^*$, let \mathbb{P}_q denote the path with vertices $0, 1, \dots, q$ linked consecutively, with a loop at 0. For a graph G , the q -th cone $M_q(G)$ (or q -th generalised Mycielskian) over G is the graph $(G \times \mathbb{P}_q) / \sim_q$, where \sim_q is the equivalence which identifies all vertices whose second coordinate is q . The vertex $(V(G) \times \{q\}) / \sim_q$ is called the *apex* of $M_q(G)$, while $V(G) \times \{0\}$ is the *base* of $M_q(G)$. Any set $V(G) \times \{i\}$ is called a *level* of $M_q(G)$.

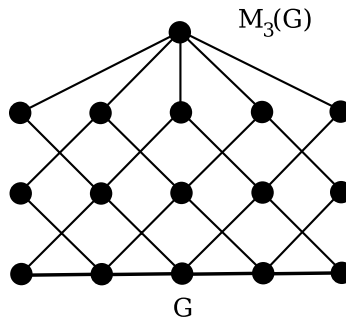


FIGURE 1. Generalised Mycielskian

The cone construction allows us to define classes of “generalised Mycielski graphs” inductively: Let $\mathcal{K}_2 = \{K_2\}$, and for $k \geq 3$, put

$$\mathcal{K}_k = \{M_q(G) : G \in \mathcal{M}_{k-1}, q \in \mathbb{N}^*\}.$$

Lemma 3 ([7]). *For any graph H , $\text{coind}(\text{Hom}(K_2, H))$ is the largest k such that there exist a $G \in \mathcal{K}_{k+2}$ admitting a homomorphism to H .*

The complex used in [7] was the box complex $B(H)$ rather than the hom-complex $\text{Hom}(K_2, H)$. However the two complexes are \mathbb{Z}_2 -homotopy equivalent by a result of Csorba [3].

3. PROOF OF THEOREM 1

3.1. Overview of the proof. One implication of Theorem 1 has been proved in [7]:

Lemma 4 (Proposition 5 of [7]). *If $\text{coind}(\text{Hom}(K_2, H)) \geq 2$, then there exists an odd cycle C_n and a homomorphism $f : C_n \rightarrow H$ such that $\sigma_1(f) = 1$.*

Proof. For reference it is worthwhile to give a sketch of the proof here. In view of Lemma 3, our hypothesis implies the existence of a homomorphism $g : M_q(C_n) \rightarrow H$. We will write $u_{i,j}$ for $g((i,j)/\sim_q)$. (Note that $(i,j)/\sim_q = \{(i,j)\}$ except when $j = q$; $u_{i,q}$ is the image of the apex of $M_q(C_n)$ for any $i \in \mathbb{Z}_n$.) We will show that the homomorphism $f : C_n \rightarrow H$ defined by $f(i) = u_{i,0}$ satisfies $\sigma_1(f) = 1$.

Consider the expressions $L_j, R_j, j = 0, \dots, q-1$ in $\mathcal{G}_{V_1}(H)$ given by

$$\begin{aligned} L_j &= (u_{0,j}, u_{1,j+1})(u_{2,j}, u_{1,j+1})^{-1}(u_{2,j}, u_{3,j+1})(u_{4,j}, u_{3,j+1})^{-1} \cdots (u_{n-1,j}, u_{0,j+1}), \\ R_j &= (u_{0,j+1}, u_{1,j})^{-1}(u_{2,j+1}, u_{1,j})(u_{2,j+1}, u_{3,j})^{-1}(u_{4,j}, u_{3,j+1}) \cdots (u_{n-1,j}, u_{0,j+1}). \end{aligned}$$

We then have

$$L_j R_j = \prod_{i=0}^{n-1} (u_{2i,j}, u_{2i+1,j+1})(u_{2i+2,j}, u_{2i+1,j+1})^{-1}, j = 0, \dots, q-1.$$

In particular, $L_{q-1} R_{q-1}$ simplifies to 1, since $u_{2i+1,q}$ is the constant image of the apex of $M_q(C_n)$. Also, by definition of θ , for $j = 1, \dots, q-1$, we have

$$(u_{2i,j}, u_{2i+1,j+1})(u_{2i+2,j}, u_{2i+1,j+1})^{-1} = (u_{2i,j}, u_{2i+1,j-1})(u_{2i+2,j}, u_{2i+1,j-1})^{-1}$$

for $i = 0, \dots, n-1$. Therefore

$$L_j R_j = \prod_{i=0}^{n-1} (u_{2i,j}, u_{2i+1,j-1})(u_{2i+2,j}, u_{2i+1,j-1})^{-1} j = 0, \dots, q-1.$$

This is the image $\phi(R_{j-1} L_{j-1})$ of $R_{j-1} L_{j-1}$ under the (well-defined) automorphism ϕ of $\mathcal{G}_{V_1}(H)$ which interchanges (x, y) with $(x, y)^{-1}$ for all $(x, y) \in A(H)$. Therefore if $L_j R_j = 1$, then $R_{j-1} L_{j-1} = 1$ and $L_{j-1} R_{j-1} = 1$, since $L_{j-1} R_{j-1}$ and $R_{j-1} L_{j-1}$ are conjugates. Therefore, $L_j R_j = 1$ for all $j = 0, \dots, q-1$. Again by definition of θ , we then have

$$\begin{aligned} \sigma_1(f) &= \prod_{i=0}^{n-1} (u_{2i,0}, u_{2i+1,0})(u_{2i+2,0}, u_{2i+1,0})^{-1} \\ &= \prod_{i=0}^{n-1} (u_{2i,0}, u_{2i+1,1})(u_{2i+2,0}, u_{2i+1,1})^{-1} = L_0 R_0 = 1. \end{aligned}$$

□

It would be nice to be able to reverse the arguments of this proof to prove the second direction. That is, start with $f : C_n \rightarrow H$ such that $\sigma_1(f) = 1$, and use the definition of θ to extend it to a homomorphism $g : M_q(C_n) \rightarrow H$. However an example in [7] shows that this is not always possible; C_n may be too small as a base.

Thus we need to start from the basic information provided by the equation $\sigma_1(f) = 1$. Since $\mathcal{G}_{\mathcal{V}_1}(H) = \mathcal{F}_{\mathcal{V}}(A(H))/\theta$ this means that in $\mathcal{F}_{\mathcal{V}_1}$, $\sigma_1(f)$ is equal to a product $\prod_{i=1}^k \gamma_i^{-1} \rho_i \gamma_i$ of conjugates of the relations ρ_i defining θ .

The first difficulty here is that there is a useful feature common to $\sigma_1(f)$ and the generators ρ_i of θ , which is lost in the expression $\prod_{i=1}^k \gamma_i \rho_i \gamma_i^{-1}$: the fact that words alternate in forward arcs and inverses of backward arcs along a walk in H . This feature provides a natural connection between algebraic expressions and path homomorphisms. For this reason, we will fix a root vertex r in H , and associate to each arc (x, y) a closed walk from r through (x, y) , called a “loop”. The use of loops will transform $\prod_{i=1}^k \gamma_i \rho_i \gamma_i^{-1}$ into a word which is much longer, but which has the desired alternating property. We will identify this word with an expression of the type $L_j R_j$ defining homomorphic images of two consecutive levels of some $M_q(C_{2m+1})$.

We will then extend this homomorphism to the apex of $M_q(C_{2m+1})$ using the definition of θ , and towards its base using the simplification of $\prod_{i=1}^k \gamma_i \rho_i \gamma_i^{-1}$ to $\sigma_1(f)$. Our basic tool to convert algebraic simplifications to homomorphism extensions is the extension along “bricks”, that is, essentially rectangular pieces that dissect $M_q(C_{2m+1})$.

The last phase of the extension will be the connection to the base of $M_q(C_{2m+1})$, which is equivalent to finding an extension that is equal on two consecutive levels.

3.2. Loops. Most of our work will be done in the free monoid $(A(H) \cup A(H)^{-1})^*$ generated by $A(H) \cup A(H)^{-1}$, where $A(H)^{-1} = \{(u, v)^{-1} : (u, v) \in A(H)\}$ is a set of symbols disjoint from $A(H)$. Of course, $\mathcal{F}_{\mathcal{V}_1}(A(H)) = (A(H) \cup A(H)^{-1})^*/\iota$, where ι is the congruence which identifies $(u, v)(u, v)^{-1}$ and $(u, v)^{-1}(u, v)$ to 1 for all $(u, v) \in A(H)$. However, for a suitable correspondence between words and walks, it is sometimes useful to avoid this identification.

A walk u_0, u_1, \dots, u_n in H is the image of a homomorphism f of some path with vertices $0, 1, \dots, n$ linked consecutively. To such a walk we can associate a word

$$\omega(f) = (u_0, u_1)(u_2, u_1)^{-1}(u_2, u_3)(u_4, u_3)^{-1} \dots$$

ending in (u_{n-1}, u_n) or $(u_n, u_{n-1})^{-1}$ depending on whether n is odd or even. This word alternates symbols from $A(H)$ and symbols from $A(H)^{-1}$, with the symbol following (u_{2i}, u_{2i+1}) being $(u_{2i+2}, u_{2i+1})^{-1}$ for some $u_{2i+2} \in V(H)$, and the symbol following $(u_{2i}, u_{2i-1})^{-1}$ being (u_{2i}, u_{2i+1}) for some $u_{2i+1} \in V(H)$. Conversely, a word with these properties naturally corresponds to a walk in H .

Now for our purposes we can assume that H is connected and nonbipartite, if necessary by restricting our attention to the component of H that contains an odd cycle with trivial signature. We fix a root vertex r and for every vertex u of H , we fix an even path $p_e(u)$ and an odd path $p_o(u)$, both from r to u . For an arc (u, v) of H , we define the loop $L(u, v) \in (A(H) \cup A(H)^{-1})^*$ corresponding to (u, v) by

$$L(u, v) = \omega(p_e(u)) \cdot (u, v) \cdot \omega(p_o(v))^{-1}.$$

Of course, $(a_1 a_2 \cdots a_n)^{-1}$ means $a_n^{-1} a_{n-1}^{-1} \cdots a_1^{-1}$, though this needs to be stated formally since inversion does not exist in $(A(H) \cup A(H)^{-1})^*$. With this notation, we define $L((u, v)^{-1})$ as $(L(u, v))^{-1}$.

The loop function naturally extends to $(A(H) \cup A(H)^{-1})^*$ by putting $L(a_1 \cdots a_n) = L(a_1) \cdots L(a_n)$. The map $L : (A(H) \cup A(H)^{-1})^* \rightarrow (A(H) \cup A(H)^{-1})^*$ is an endomorphism whose image consists of words corresponding to even closed walks rooted at r .

3.3. Cycles with trivial signature. Now let $C_{n'}$ be an odd cycle and $f : C_{n'} \rightarrow H$ a homomorphism such that $\sigma_1(f) = 1$. For our purposes, it is useful to assume that for $i \in \mathbb{Z}_{n'} = V(C_{n'})$, we have $f(i-1) \neq f(i+1)$. This can be done without loss of generality, since if $f(i-1) = f(i+1)$, then we can remove i and identify $i-1$ and $i+1$ to create a copy of $C_{n'-2}$ on which f induces a homomorphism $f' : C_{n'-2} \rightarrow H$. We then have $\sigma_1(f') = \sigma_1(f)$, since $\sigma_1(f')$ is obtained from $\sigma_1(f)$ by cancelling out $(f(i-1), f(i))$ with $(f(i+1), f(i))^{-1}$ and $(f(i), f(i-1))^{-1}$ with $(f(i), f(i+1))$ (unless $i = 0$, in which case we get a conjugation of the identity element instead of the second cancellation).

We begin by lifting the condition $\sigma_1(f) = 1$ from $\mathcal{G}(H)$ to $\mathcal{F}_{V_1}(A(H))$: by definition, this means that there exists a sequence

$$\{\rho_i = (a_i, b_i)(c_i, b_i)^{-1}(c_i, d_i)(a_i, d_i)^{-1} : i = 1, \dots, k\}$$

of generators of θ and a sequence $\gamma_1, \dots, \gamma_k$ of conjugating elements such that

$$(2) \quad \prod_{i=0}^{n'-1} (f(2i), f(2i+1)) \cdot (f(2i+2), f(2i+1))^{-1} = \prod_{i=1}^k \gamma_i \cdot \rho_i \cdot \gamma_i^{-1}.$$

Note that the left side is reduced in $\mathcal{F}_{V_1}(A(H))$, since $f(j-1) \neq f(j+1)$ for all $j \in \mathbb{Z}_{n'}$. This means that the right side simplifies to the left, by repeatedly cancelling out adjacent terms that are inverse of each other.

We view the terms in this equation as elements of $(A(H) \cup A(H)^{-1})^*$ (that is, as their simplest preimage under ι). In this way the terms $L(\rho_i)$ and $L(\gamma_i)$ are well defined. Now consider the word

$$w' = \prod_{i=1}^k L(\gamma_i) \cdot L(\rho_i) \cdot L(\gamma_i^{-1}).$$

Its length is a multiple of 4, since for each i , the length of $L(\gamma_i) \cdot L(\rho_i) \cdot L(\gamma_i^{-1})$ is a multiple of 4. Therefore for any neighbour s of r , the word $w = w' \cdot (r, s) \cdot (r, s)^{-1}$ has length equal to $2n$ for some odd n . We then have $w = \omega(g)$ for some closed walk g . We identify the domain of g with the $2n$ -cycle $C_n \times K_2$, whose vertices correspond to two consecutive levels of some $M_q(C_n)$. We will show such that it is possible to choose q and the correspondence such that g extends to a homomorphism of $M_q(C_n)$ to H .

3.4. Bricks and simplifications. For integers i and j , the bricks $B_S(i, j)$ and $B_D(i, j)$ are the graphs defined by

$$\begin{aligned} V(B_S(i, j)) &= \{(x, y) : 0 \leq x \leq i, 0 \leq y \leq j, \text{ } x \text{ and } y \text{ have the same parity}\}, \\ E(B_S(i, j)) &= \{[(x, y), (x', y')] : |x - x'| = 1, |y - y'| = 1\}; \\ V(B_D(i, j)) &= \{(x, y) : 0 \leq x \leq i, 0 \leq y \leq j, \text{ } x \text{ and } y \text{ have different parities}\}, \\ E(B_D(i, j)) &= \{[(x, y), (x', y')] : |x - x'| = 1, |y - y'| = 1\}; \end{aligned}$$

Thus, $B_S(i, j)$ and $B_D(i, j)$ are the two connected components of the categorical product of paths of lengths i and j . The sets of vertices of $B_S(i, j)$ and $B_D(i, j)$ with second coordinate 0 are called the *lower side* of $B_S(i, j)$ and $B_D(i, j)$, and similarly, their *upper*, *left*, and *right* sides are defined by obvious conditions.

For any i , $B_S(2i, 1)$ and $B_D(2i, 1)$ are paths of length $2i$. We use the following extension properties of their homomorphisms to H .

Lemma 5. *Let $h : B_S(2i, 1) \rightarrow V(H)$, $h' : B_D(2i, 1) \rightarrow V(H)$ be homomorphisms.*

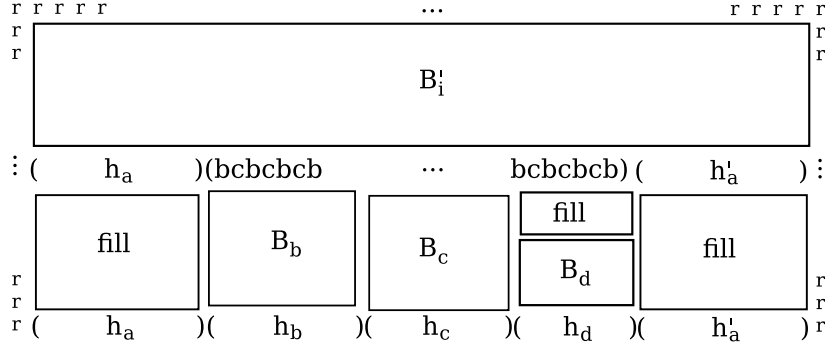
- (i) *For any $j \geq 0$, the map \hat{h} defined by $\hat{h}(x, y) = h(x, (y \bmod 2))$ is a homomorphism of $B_S(2i, j)$ to H . Similarly, for any $j \geq 0$, the map \hat{h}' defined by $\hat{h}'(x, y) = h'(x, 1 - (y \bmod 2))$ is a homomorphism of $B_D(2i, j)$ to H .*
- (ii) *If $h(x, x \bmod 2) = h(2i - x, x \bmod 2)$ for $x = 0, \dots, i$, then there exists a homomorphism $\hat{h} : B_S(2i, 2\lceil i/2 \rceil) \rightarrow H$ extending h such that \hat{h} is identically equal to $h(0)$ on the left, upper and right sides of $B_S(2i, 2\lceil i/2 \rceil)$. Similarly, if $h'(x, 1 - (x \bmod 2)) = h'(2i - x, 1 - (x \bmod 2))$ for $x = 0, \dots, i$, then there exists a homomorphism $\hat{h}' : B_D(2i, 2\lceil i/2 \rceil + 1) \rightarrow H$ extending h' such that \hat{h}' is identically equal to $h(0, 1)$ on the left, upper and right sides of $B_D(2i, 2\lceil i/2 \rceil + 1)$.*

Proof. Item (i) is straightforward. To prove the first part of item (ii), we note that the distance between two vertices (x, y) , (x', y') of $B_S(2i, j)$ is the “bus distance” $\max\{|x - x'|, |y - y'|\}$. Therefore the map $\hat{h} : B_S(2i, 2\lceil i/2 \rceil) \rightarrow H$ defined by $\hat{h}(x, y) = h(x', y')$, where $(x', y') \in B_S(2i, 1)$ is at the same distance as (x, y) from $(i, (i \bmod 2))$, is a homomorphism with the prescribed properties. The second part is proved similarly. \square

3.5. Extension to the apex. We apply Lemma 5 to extend the homomorphism $g : C_n \times K_2 \rightarrow H$ towards the apex of some $M_q(C_n)$. Consider the restrictions g_1, \dots, g_k of g such that $\omega(g_i) = L(\gamma_i) \cdot L(\rho_i) \cdot L(\gamma_i^{-1})$. For a fixed i , we further decompose g_i into five restrictions h_a, h_b, h_c, h_d, h'_a such that

$$\begin{aligned} \omega(h_a) &= L(\gamma_i) \cdot \omega(p_e(a_i)), \\ \omega(h_b) &= \omega(p_o(b_i))^{-1} \circ \omega(p_o(b_i)), \\ \omega(h_c) &= \omega(p_e(c_i))^{-1} \circ \omega(p_e(c_i)), \\ \omega(h_d) &= \omega(p_o(d_i))^{-1} \circ \omega(p_o(d_i)), \\ \omega(h'_a) &= \omega(p_e(a_i))^{-1} \cdot L(\gamma_i). \end{aligned}$$

By Lemma 5 (ii), h_c extends to a homomorphism \hat{h}_c of some $B_c = B_S(2\ell, \ell)$ to H (ℓ being the length of $p_e(c_i)$), which is identically equal to c_i on its left, upper and right sides. Similarly h_b and h_d extend to \hat{h}_b and \hat{h}_d , which are identically

FIGURE 2. Extension of g_i

equal to b_i and d_i respectively on their left, upper and right sides. Moreover, the extensions $\hat{h}_b, \hat{h}_c, \hat{h}_d$ of h_b, h_c, h_d to B_b, B_c, B_d can be carried simultaneously side by side, since c_i is adjacent to b_i and d_i .

We then add one level to B_c and extend \hat{h}_c by mapping the new level to b_i , and add two levels to each of B_b and B_d and extend \hat{h}_b and \hat{h}_d by mapping the new levels to c_i and b_i respectively. We then use Lemma 5 (i) on the top two levels of Q_b, Q_c and Q_d to add levels and equalize their heights if necessary. We can then use Lemma 5 (i) to extend h_a and h'_a upwards to match the height of B_b, B_c and B_d .

We have thus extended g_i to $\hat{g}_i : B_i \rightarrow H$ such that the restriction g'_i of \hat{g}_i to the top two levels of Q_i satisfies

$$\omega(g'_i) = L(\gamma_i) \circ \omega(p_e(a_i)) \circ (a_i, b_i) \circ ((c_i, b_i)^{-1} (c_i, b_i))^{e_i} (a_i, b_i)^{-1} \circ \omega(p_e(a_i))^{-1} \circ L(\gamma_i)^{-1},$$

for some e_i (see Figure 2). Using Lemma 5 (ii), we extend g'_i to $\hat{g}'_i : B'_i \rightarrow H$ which is identically r on the left, top and right sides of B'_i .

These extensions of g_i to \hat{g}_i and \hat{g}'_i can be carried simultaneously for $i = 1, \dots, k$, since they all have the value r at their common boundaries. We can then use Lemma 5 (i) to equalize heights, and bring up the part of g corresponding to $(r, s)(r, s)^{-1}$ to the same height. Since the value of the extensions is identically r at the top level, we can identify all the vertices of this top level. We have extended g to the apex of some $M_q(C_n)$.

3.6. Extension towards the base. The extension of g towards the base of $M_q(C_n)$ is a second extension independent from the extension to the apex. It again uses Lemma 5, thus we keep the terminology of extending in the “upper” direction. The two extensions will afterwards be merged together by identifying their bottom level.

In $\mathcal{F}_{V_1}(A(H))$, the word $\prod_{i=1}^k \gamma_i \cdot \rho_i \cdot \gamma_i^{-1}$ of $(A(H) \cup A(H)^{-1})^*$ simplifies to

$$\sigma_1(f) = \prod_{i=0}^{n'-1} (f(2i), f(2i+1)) \cdot (f(2i+2), f(2i+1))^{-1}.$$

Specifically, this means that there exists a sequence of basic simplifications of $\prod_{i=1}^k \gamma_i \cdot \rho_i \cdot \gamma_i^{-1}$ eliminating everything but the terms of $\sigma_1(f)$. In each of these basic simplification, some term $t_i = x$ will cancel out either with the next term

$t_{i+1} = x^{-1}$, or with a further term $t_{i+2j} = x^{-1}$, where all the intermediate terms $t_{i+1}, \dots, t_{i+2j-1}$ have been previously simplified.

In $\prod_{i=1}^k L(\gamma_i \cdot \rho_i \cdot \gamma_i^{-1})$, a term t_i is replaced by $L(t_i)$. Consider the homomorphism corresponding to the word $L(t_i) \circ ((r, s)(r, s)^{-1})^{e_i} \circ L(t_i^{-1})$ for some e_i . By Lemma 5 (ii), it extends to a homomorphism of some B_i to H with value identically r on its left, upper and right sides. We can then add a level with value identically s , and use Lemma 5 (i) to bring up the remainder of the extension of g to the same level.

In this way, we match each step in the simplification of $\prod_{i=1}^k \gamma_i \cdot \rho_i \cdot \gamma_i^{-1}$ to a corresponding step in the extension. We end up with an extension where the two upper levels correspond to a word with the terms $L(f(2i), f(2i+1))$ and $L(f(2i+2), f(2i+1))^{-1}$ separated by terms of the form $((r, s)(r, s)^{-1})^e$.

Now in this word, between $(f(2i), f(2i-1))^{-1}$ and $(f(2i), f(2i+1))$ is a word of the form $\omega(p_e(f(2i))^{-1} \circ ((r, s)(r, s)^{-1})^e \circ \omega(p_e(f(2i)))$. By Lemma 5 (ii), it extends to a homomorphism of some brick to H with value identically $f(2i)$ on its left, upper and right sides. We can then add a level with value identically $f(2i-1)$. Similarly between $(f(2i), f(2i+1))$ and $(f(2i+2), f(2i+1))^{-1}$ is a word of the form $\omega(p_o(f(2i+1))^{-1} \circ ((r, s)(r, s)^{-1})^e \circ \omega(p_o(f(2i+1)))$. By Lemma 5 (ii), it extends to a homomorphism of some brick to H with value identically $f(2i+1)$ on its left, upper and right sides. We can then add a level with value identically $f(2i)$.

These extensions can be carried out side by side simultaneously, since $f(j)$ is adjacent to $f(j+1)$ for all j . We use Lemma 5 (i) to equalize the height. We now have extended g so that (after a suitable cyclic shift), the homomorphism on the top two levels corresponds to a word of the form

$$\prod_{i=0}^{n'-1} [((f(2i), f(2i+1))(f(2i), f(2i+1))^{-1})^{e_i}(f(2i), f(2i+1)) \cdot ((f(2i+2), f(2i+1))^{-1}(f(2i+2), f(2i+1)))^{e_i}(f(2i+2), f(2i+1))^{-1}].$$

3.7. Connection to the base. Let $f' : C_n \times K_2 \rightarrow H$ be the homomorphism corresponding to the top two levels of the second extension of g . Note that f' follows the original f twice around its image, with every arc traced back and forth many times. More precisely, $f' = h \circ f$, where $h : C_n \times K_2 \rightarrow C_{n'}$ is a homomorphism. The only thing missing is to have $h(i, 0) = h(i, 1)$ for all i , that is, to have f' equal on the two levels. These levels could then be identified to form the base of $M_q(C_n)$.

We label the vertices on $C_n \times K_2 = C_{2n}$ consecutively u_0, \dots, u_{2n-1} (with indices in \mathbb{Z}_{2n}). We label the edge $[u_i, u_{i+1}]$ with the sign $+$ (resp. $-$) if for some $j \in \mathbb{Z}_{n'}$ we have $h(u_i) = j$ and $h(u_{i+1}) = j+1$ (resp $h(u_{i+1}) = j-1$). The word $\omega(f')$ above shows that the number of edges with label $+$ is $n+n'$, and the number of edges with label $-$ is $n-n'$.

Opposite signs on two consecutive edges $[u_i, u_{i+1}], [u_{i+1}, u_{i+2}]$ happen precisely when $h(u_i) = h(u_{i+2})$. We then have $h(u_{i+1}) \in \{h(u_i) - 1, h(u_i) + 1\}$. Substituting one value for the other will interchange the signs of $[u_i, u_{i+1}]$ and $[u_{i+1}, u_{i+2}]$. In terms of the extension of g , this corresponds to adding two levels to match the operation. That is, the extension on the two new levels is identical to the extension on the previous two levels, except that value at the vertex corresponding to u_{i+1} switches from one term in $\{f(h(u_i) - 1), f(h(u_i) + 1)\}$ to the other.

Proceeding this way, we can move the labels around in any way we please. In particular we can rearrange the labels until there are $(n+n')/2$ “+” labels followed by $(n-n')/2$ “-” labels, then by $(n+n')/2$ “+” labels and $(n-n')/2$ “-” labels.

In this way, between any u_i and u_{i+n} , the value of h moves forward $(n+n')/2$ times and backward $(n-n')/2$ times so that it ends up n' places forward in $C_{n'}$, exactly where it started. Our second extension of g is then equal on the top two levels, so that these levels can be identified to form the base of $M_q(C_n)$. This concludes the proof of Theorem 1. ■

4. PROOF OF THEOREM 2

4.1. Algorithmic considerations. To prove Theorem 2, we will show that the existence of an odd cycle in H with zero signature is incompatible with the existence of a \mathbb{Z}_2 -map of some $S^m(\text{Hom}(K_2, H))$ to Q_1 . Note that Hom and S are both exponential constructions. We will first show that the detection of an odd cycle with zero signature can be done efficiently in terms of the size of H .

To each arc (u, v) of H we associate a variable $X_{u,v} \in \mathbb{Z}_2$. We consider the system consting of the following equations.

- The *flow constraint* at a vertex u of H is the equation

$$\sum_{v \in N_H(u)} (X_{u,v} + X_{v,u}) = 0.$$

(Where $N_H(u)$ is the neighbourhood of u in H .)

- The *parity constraint* is the global condition

$$\sum_{(u,v) \in A(H)} X_{u,v} = 1.$$

- The *signature constraint* is the equation

$$\sum_{(u,v) \in A(H)} ((X_{u,v} - X_{v,u}) \cdot (u, v)) = 0.$$

The flow and parity constraints are equations in \mathbb{Z}_2 . The signature constraint is a single equation in $\mathcal{G}_{\mathbb{Z}_2}(H)$. The group $\mathcal{G}_{\mathbb{Z}_2}(H)$ has a minimal generating set with no more than $|A(H)|$ elements. Each arc (u, v) can be expressed in terms of this generating set. The signature constraint is then the set of constraints corresponding to voiding the coefficient of each element of this generating set in the expression $\sum_{(u,v) \in A(H)} (X_{u,v} - X_{v,u}) \cdot (u, v)$. Thus the system has no more than $|V(H)| + 1 + |A(H)|$ linear equations in $|A(H)|$ variables.

If for some odd cycle C_n there is a homomorphism $f : C_n \rightarrow H$ such that $\sigma_2(f) = 0$, then the system above has a solution, obtained by putting $X_{u,v} = 1$ if there is an odd number of elements i of \mathbb{Z}_n such that $f(i) = u$ and $f(i+1) = v$, and $X_{u,v} = 0$ otherwise. Conversely, for each solution of the system, the subdigraph of H spanned by the arcs (u, v) such that $X_{u,v} = 1$ is Eulerian, though not necessarily connected. We can modify the solution by giving the value 1 to variables $X_{u,v}$, $X_{v,u}$ corresponding to opposite arcs, to make the subdigraph connected. Indeed it is clear that this modification does not alter the validity of the solution. An Euler tour of the subdigraph then corresponds to a homomorphism $f : C_n \rightarrow H$ with n odd, such that $\sigma_2(f) = 0$. Thus, the existence of some odd cycle with zero signature can be detected in polynomial time.

4.2. \mathbb{Z}_2 -maps of crown. Most of our work will involve groups of the form $\mathbb{Z}_2^{P^2}$, where P is a poset. Here, $\underline{2}$ is the poset with elements $0, 1$ such that $0 < 1$. For a poset P , P^2 is the set of all order-preserving maps of $\underline{2}$ to P , including the constant maps. We will represent an element of P^2 by the comparability $(x \leq y)$ it represents. $\mathbb{Z}_2^{P^2}$ is the 2-group generated by P^2 . To an order preserving-map $f : P \rightarrow Q$, we naturally associate the group homomorphism $\hat{f} : \mathbb{Z}_2^{P^2} \rightarrow \mathbb{Z}_2^{Q^2}$ which extends the map between generators defined by f .

A \mathbb{Z}_2 -crown is a \mathbb{Z}_2 -poset P with elements $\pm 0, \dots, \pm(2n-1)$ and the relations

$$0 < 1 > 2 < 3 > \dots < 2n-1 > -0 < -1 > -2 < -3 > \dots < -(2n-1) > 0.$$

The *strict order indicator* $1_P^<$ on P is the element of $\mathbb{Z}_2^{P^2}$ with value 1 on injective maps and 0 on constant maps. Our argument is partly based on the following result, which is a simplicial statement of the fact that an antipodal continuous self-map of the circle has “odd degree”.

Lemma 6. *Let P be a \mathbb{Z}_2 -crown and $f : P \rightarrow Q_1$ a \mathbb{Z}_2 -map. Then $\hat{f}(1_P^<) = 1_{Q_1}^<$.*

Proof. Each connected component of $f^{-1}(+0)$ and of $f^{-1}(-0)$ starts and ends in an even number, and each connected component of $f^{-1}(+1)$ and of $f^{-1}(-1)$ starts and ends in an odd number. Thus the coefficient of each of $(+0, +0)$, $(-0, -0)$, $(+1, +1)$, $(-1, -1)$ in $\hat{f}(1_P^<)$ is 0. Now suppose without loss of generality that $f(0) = 0$ and $f(-0) = -0$. Then on the “positive zig-zag” $0 < 1 > 2 < 3 > \dots < 2n-1 > -0$, the image of f will switch from $+0$ to -0 some n times, and switch back from -0 to $+0$ $n-1$ times. Every switch goes through $+1$ or -1 , so one of the pairs $(+0, +1)$, $(-0, +1)$ or $(+0, -1)$, $(-0, -1)$ of strict comparabilities is touched an odd number of times, and the other an even number of times. Since f is a \mathbb{Z}_2 -map, these numbers are reversed on the “negative zig-zag” $-0 < -1 > -2 < -3 > \dots < -2n+1 > 0$, so that each of the strict comparabilities $(+0, +1)$, $(-0, +1)$, $(+0, -1)$, $(-0, -1)$ is touched an odd number of times. Thus $\hat{f}(1_P^<) = 1_{Q_1}^<$. \square

4.3. Cycles with null signature. A graph homomorphism $g : C_n \rightarrow H$ induces the \mathbb{Z}_2 -map $g' : \text{Hom}(K_2, C_n) \rightarrow \text{Hom}(K_2, H)$ defined by $g'(A, B) = (g(A), g(B))$. However, it will be useful to associate to g a different map $g^+ : \text{Hom}(K_2, C) \rightarrow \text{Hom}(K_2, H)$ defined as follows.

- A minimal element $(\{i\}, \{j\})$ of $\text{Hom}(K_2, C_n)$ correspond to an arc $(i, j) = (i, i \pm 1)$ of C , and we put $g^+(\{i\}, \{j\}) = (\{f(i)\}, \{f(j)\})$, its natural image induced by g .
- For a maximal element of the form $(\{i\}, \{i-1, i+1\})$, we put $g^+(\{i\}, \{i-1, i+1\}) = (\{g(i)\}, N_H(g(i)))$. Similarly, for a maximal element of the form $(\{i-1, i+1\}, \{i\})$, we put $g^+(\{i-1, i+1\}, \{i\}) = (N_H(g(i)), \{g(i)\})$.

Thus if $f : \text{Hom}(K_2, H) \rightarrow Q_1$ is a \mathbb{Z}_2 -map, then $\widehat{f \circ g^+}(1_{\text{Hom}(K_2, C)}^<) = 1_{Q_1}^<$ by Lemma 6. The same holds with barycentric subdivisions: $S^m(\text{Hom}(K_2, C))$ is a \mathbb{Z}_2 -crown, and if $f : S^m(\text{Hom}(K_2, H)) \rightarrow Q_1$ is a \mathbb{Z}_2 -map, then $f \circ \widehat{S^m(g^+)}(1_{S^m(\text{Hom}(K_2, C))}^<) = 1_{Q_1}^<$.

Now, if $\sigma_2(g) = 0$, then in $\mathcal{F}_{\mathbb{Z}_2}(A(H))$ we have

$$(3) \quad \sigma_2(g) = \sum_{i=0}^{n-1} [(g(2i), g(2i+1)) + (g(2i+2), g(2i+1))] = \sum_{j=1}^k \rho_j,$$

where ρ_1, \dots, ρ_k are relations of the form $\rho_j = (a_j, b_j) + (c_j, b_j) + (c_j, d_j) + (a_j, d_j)$ defining the congruence θ on $\mathcal{F}_{V_2}(A(H))$. To $\rho = (a, b) + (c, b) + (c, d) + (a, d)$, we associate the subposet ρ^+ of $\text{Hom}(K_2, H)$ induced by the set $\{(\{a\}, N_H(a)), (N_H(b), \{b\}), (\{c\}, N_H(c)), (N_H(d), \{d\}), (\{a\}, \{b\}), (\{c\}, \{b\}), (\{c\}, \{d\}), (\{a\}, \{d\})\}$. (See Figure 3.)

Our next Lemma adapts Equation (3) to the groups $\mathbb{Z}_2^{\text{Hom}(K_2, H)^2}$ and $\mathbb{Z}_2^{\text{Hom}(K_2, S^m(H))^2}$ for all $m \geq 1$. Recall that $g^+ : \text{Hom}(K_2, C_n) \rightarrow \text{Hom}(K_2, H)$ induces $\widehat{g^+} : \mathbb{Z}_2^{\text{Hom}(K_2, C_n)^2} \rightarrow \mathbb{Z}_2^{\text{Hom}(K_2, H)^2}$; g^+ also induces $S^m(g^+) : S^m(\text{Hom}(K_2, C_n)) \rightarrow S^m(\text{Hom}(K_2, H))$ for all $m \geq 1$, which in turn induce $\widehat{S^m(g^+)} : \mathbb{Z}_2^{S^m(\text{Hom}(K_2, C_n))^2} \rightarrow \mathbb{Z}_2^{S^m(\text{Hom}(K_2, H))^2}$.

Lemma 7.

$$\widehat{g^+}(1_{\text{Hom}(K_2, C_n)}^<) = \sum_{j=1}^k 1_{\rho_j^+}^< ,$$

hence

$$\widehat{S^m(g)}(1_{S^m(\text{Hom}(K_2, C_n))}^<) = \sum_{j=1}^k 1_{S^m(\rho_j^+)}^<$$

for all $m \geq 1$.

Proof. Let O_g be the set of arcs of H which appear an odd number of times as $(g(i), g(i+1))$ or $(g(i+1), g(i))$ for some $i \in \mathbb{Z}_n$. Thus in $\mathcal{F}_{V_2}(A(H))$, we have $\sigma_2(g) = \sum_{(u,v) \in O_g} (u, v)$. By Equation 3, O_g coincides with the set of arcs which appear an odd number of times as terms in ρ_1, \dots, ρ_k .

Now for $(u, v) \in A(H)$, let $V(u, v)$ be the subposet

$$(N_H(v), \{v\}) > (\{u\}, \{v\}) < (\{u\}, N_H(u))$$

of $\text{Hom}(K_2, H)$. Then

$$\widehat{g^+}(1_{\text{Hom}(K_2, C_n)}^<) = \sum_{(u,v) \in O_g} 1_{V(u,v)}^< ,$$

and similarly for all

$$\widehat{S^m(g)}(1_{S^m(\text{Hom}(K_2, C))}^<) = \sum_{(u,v) \in O_g} 1_{S^m(V(u,v))}^<$$

for all $m \geq 1$. Equation 3 then implies that

$$\sum_{(u,v) \in O_g} 1_{V(u,v)}^< = \sum_{j=1}^k 1_{\rho_j^+}^<$$

and

$$\sum_{(u,v) \in O_g} 1_{S^m(V(u,v))}^< = \sum_{j=1}^k 1_{S^m(\rho_j^+)}^<$$

for all $m \geq 1$. □

We will next see that the conclusion

$$\widehat{S^m(g)}(1_{S^m(\text{Hom}(K_2, C))}^<) = \sum_{j=1}^k 1_{S^m(\rho_j^+)}^<$$

of Lemma 7 is incompatible with the conclusion

$$\widehat{S^m(g)}(1_{S^m(\text{Hom}(K_2, C))}^<) = 1_{Q_1}^<$$

of Lemma 6.

4.4. Domination and dismantlability. In a poset Q , an element p is said to be *dominated* by an element q if p is comparable to q and every element comparable to p is comparable in the same way to q . Q is said to be *dismantlable* if it can be reduced to a single point by recursively removing dominated elements.

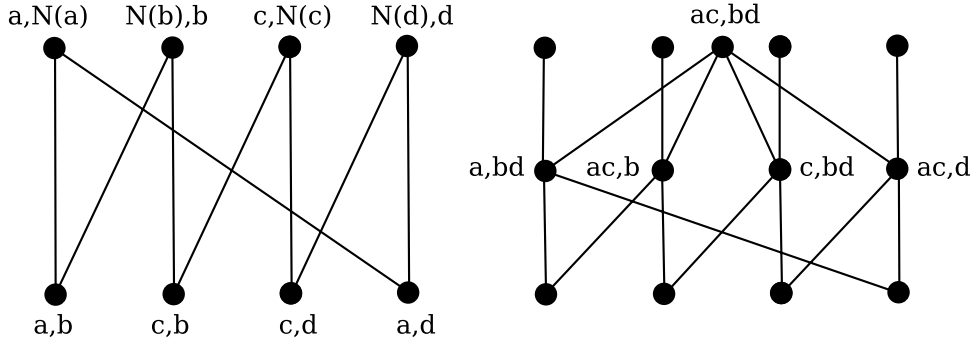


FIGURE 3. ρ^+ and $D(\rho^+)$

For instance, consider the subposet $D(\rho^+)$ of $\text{Hom}(K_2, H)$ obtained by adding the elements $(\{a\}, \{b, d\})$, $(\{a, c\}, \{b\})$, $(\{c\}, \{b, d\})$, $(\{a, c\}, \{d\})$ and $(\{a, c\}, \{b, d\})$ to ρ^+ . (See Figure 3.) In $D(\rho^+)$, $(\{a\}, N_H(a))$, $(N_H(b), \{b\})$, $(\{c\}, N_H(c))$ and $(N_H(d), \{d\})$ are dominated respectively by $(\{a\}, \{b, d\})$, $(\{a, c\}, \{b\})$, $(\{c\}, \{b, d\})$ and $(\{a, c\}, \{d\})$. Removing these dominated elements leaves $(\{a, c\}, \{b, d\})$ as the unique maximum. We then have $(\{a\}, \{b, d\})$, $(\{a, c\}, \{b\})$, $(\{c\}, \{b, d\})$ and $(\{a, c\}, \{d\})$ dominated by $(\{a, c\}, \{b, d\})$, and removing these leaves $(\{a\}, \{b\})$, $(\{c\}, \{b\})$, $(\{c\}, \{d\})$ and $(\{c\}, \{b, d\})$ dominated by $(\{a, c\}, \{b, d\})$. Hence $D(\rho^+)$ is dismantlable.

Lemma 8. *Let P be a \mathbb{Z}_2 -crown, Q a dismantlable poset and $f : P \rightarrow Q_1$ an order-preserving map which factors through Q , that is, $f = h \circ g$ where $g : P \rightarrow Q$, $h : Q \rightarrow Q_1$ are order-preserving. Then $\hat{f}(1_{\hat{P}}^<) = 0$.*

Proof. The result is clear if Q is a single point. Thus we can proceed by induction on the number of elements in Q . Let p be dominated by q in Q ; we will suppose that $p < q$ (the other case being symmetric). Let $r : Q \rightarrow Q$ be the retraction which maps p to q and fixes everything else. Then $f' = h \circ r \circ g : P \rightarrow Q_1$ factors through the dismantlable poset $r(Q)$ which has one element less than Q , so by the induction hypothesis, $\hat{f}'(1_{\hat{P}}^<) = 0$. Thus if $f = f'$, then $\hat{f}(1_{\hat{P}}^<) = 0$. We can therefore suppose that $f \neq f'$. This means that $h(p) \neq h(q)$. We will suppose without loss of generality that $h(p) = 0$ and $h(q) = 1$.

Let $g' : P \rightarrow Q_1$ be the map obtained from g by changing the image of every maximal element x such that $g(x) = p$ from p to q . For each such x , there are two minimal elements y, z of P which are below x . We then have $f(y), f(z) \leq f(x) = 0$,

so that $f(y) = f(z) = 0$. Therefore the two comparabilities $(y \leq x)$, $(z \leq x)$ are mapped to $(0 \leq 0)$ by \hat{f} and to $(0 \leq 1)$ by $\widehat{h \circ g'}$. Thus $\widehat{h \circ g'}(1_P^<) = \hat{f}(1_P^<)$.

Now $r \circ g$ is obtained from g' by changing the image of every minimal element x such that $g'(x) = p$ from 0 to 1. For each such x , there are two maximal elements y, z of P which are above x . Now since q dominates p , we have $g'(y), g'(z) \geq q$ hence $h \circ g'(y) = h \circ g'(z) = 1$. Therefore the two comparabilities $(x \leq y)$, $(x \leq z)$ are mapped to $(0 \leq 1)$ by $\widehat{h \circ g'}$ and to $(1 \leq 1)$ by $\widehat{h \circ r \circ g} = \hat{f}'$. Thus $\hat{f}'(1_P^<) = \widehat{h \circ g'}(1_P^<) = \hat{f}(1_P^<)$. Therefore $\hat{f}'(1_P^<) = 0$ implies $\hat{f}(1_P^<) = 0$. \square

Lemma 9. *If Q is a dismantlable poset, then for any m , $S^m(Q)$ is dismantlable.*

Proof. It suffices to show that if Q is dismantlable, then $S(Q)$ is dismantlable. We will again proceed by induction on the number of elements in Q , the result being clear if Q is a single point. Let p be dominated by q in Q . We will show that $S(Q)$ dismantles to $S(Q \setminus \{p\})$. The elements of $S(Q)$ are chains in Q , and since q dominates p , for every element C of $S(Q)$ containing p , $C \cup \{q\}$ is an element of $S(Q)$. Let m be the number of elements of $S(Q)$ which contain p but not q . We construct a sequence $S(Q) = R_0, R_1, \dots, R_m$ of subposets of $S(Q)$, where R_i is obtained from R_{i-1} by removing a maximal element C_i of R_{i-1} which contains p but not q . Since C_i is dominated by $C_i \cup \{q\}$ in R_{i-1} , the sequence is a dismantling of $S(Q)$ to its subposet R_m which consists of all the elements which contain q whenever they contain p . Let $R_m, R_{m+1}, \dots, R_{2m}$ be a sequence of subposets of R_m , where R_i is obtained from R_{i-1} by removing a minimal element C_i of R_{i-1} which contains p . Since C_i is dominated by $C_i \setminus \{p\}$ in R_{i-1} , the sequence is a dismantling of R_m to its subposet $R_{2m} = S(Q \setminus \{p\})$. Thus $S(Q)$ dismantles to $S(Q \setminus \{p\})$. Therefore if $S(Q \setminus \{p\})$ is dismantlable, then so is $S(Q)$. \square

Proof of Theorem 2. Suppose that for some odd n there exists a homomorphism $f : C_n \rightarrow H$ such that $\sigma_2(g) = 0$. Then there exists a sequence ρ_1, \dots, ρ_k of generators of θ such that $\sigma_2(f) = \sum_{j=1}^k \rho_j$ in $\mathcal{F}_{V_2}(A(H))$. By Lemma 7, we then have

$$\widehat{S^m(f^+)}(1_{S^m(\text{Hom}(K_2, C_n))}^<) = \sum_{j=1}^k 1_{S^m(\rho_j^+)}^<$$

for all $m \geq 0$. Now for $j = 1, \dots, k$, ρ_j^+ is contained in $D(\rho_j^+)$ which is dismantlable, hence $S^m(D(\rho_j^+))$ is dismantlable for all m by Lemma 9. Hence by Lemma 9, for any order-preserving map $g : S^m(\text{Hom}(K_2, H)) \rightarrow Q_1$, we have $\hat{g}(1_{S^m(\rho_j^+)}^<) = 0$ for $j = 1, \dots, k$. Therefore

$$g \circ \widehat{S^m(f^+)}(1_{S^m(\text{Hom}(K_2, C_n))}^<) = \sum_{j=1}^k \hat{g}(1_{S^m(\rho_j^+)}^<) = 0.$$

By Lemma 6, $g \circ S^m(f^+) : S^m(\text{Hom}(K_2, C_n)) \rightarrow Q_1$ cannot be a \mathbb{Z}_2 -map. Since $S^m(f^+) : S^m(\text{Hom}(K_2, C_n)) \rightarrow S^m(\text{Hom}(K_2, H))$ is a \mathbb{Z}_2 -map, we conclude that there does not exist a \mathbb{Z}_2 -map $g : S^m(\text{Hom}(K_2, H)) \rightarrow Q_1$. Therefore $\text{ind}(\text{Hom}(K_2, H)) \geq 2$. \square

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